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A Remark on the Low-Temperature Behavior of the SOS Interface in Half-Space

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We investigate the low-temperature phase diagram of the d-dimensional ($d \geqslant 2$) solid-on-solid (SOS) interface constrained to lie above a rigid wall to which it is attracted by a constant force. We prove uniqueness of the Gibbs state and exponentially fast convergence (in the diameter of the domain) of the finite-volume expectation of local observables, for all values of parameters where uniqueness in the class of translation-periodic limit Gibbs states was established previously. These domains of uniqueness are bordered by lines on which the system undergoes a layering transition.

KEY WORDS: Gibbs state; SOS model; layering transition; uniqueness; FKG; cluster expansion.

1. THE MODEL AND THE RESULT

This paper continues the work of refs. 4 and 3, where the phenomenon of the layering transition was studied for the model defined by the formal SOS Hamiltonian

$$H(\varphi) = \sum_{(x,x')} |\varphi_x - \varphi_{x'}| + h \sum_x \varphi_x \tag{1}$$

Here the height (spin) variable φ_x takes the values from $\mathbb{Z}^+ = \{1, 2, ...\}$, the external field h is positive, the first sum is over all pairs (x, x') of nearest neighbor sites in $\mathbb{Z}^d$ ($d \geqslant 2$), and the second sum is over all $x \in \mathbb{Z}^d$. Given

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a configuration $\varphi_V \in (\mathbb{Z}^+)^V$ in a finite domain $V \in \mathbb{Z}^d$ and a boundary condition $\bar{\varphi}_{V^c} \in (\mathbb{Z}^+)^{V^c}$, $V^c = \mathbb{Z}^d \backslash V$, the conditional Hamiltonian is

$$H(\varphi_V | \bar{\varphi}_{V^c}) = \sum_{(x,x'):\, x,x' \in V} |\varphi_x - \varphi_{x'}| + \sum_{(x,x'):\, x \in V,\, x' \in V^c} |\varphi_x - \bar{\varphi}_{x'}| + h \sum_{x \in V} \varphi_x \tag{2}$$

Here and below the sum $\sum_{(x,x'):\, x,x' \in V}$ is taken over all pairs (x, x') of nearest neighbor sites in V and the sum $\sum_{(x,x'):\, x \in V, x' \in V^c}$ is taken over all pairs (x, x') of nearest neighbor sites with one site in V and other site in V^c. The corresponding partition function is

$$\Xi(V | \bar{\varphi}_{V^c}) = \sum_{\varphi_V} \exp(-\beta H(\varphi_V | \bar{\varphi}_{V^c})) \tag{3}$$

and the finite-volume Gibbs measure is defined by

$$\langle f(\varphi_V) \rangle_{V, \bar{\varphi}_{V^c}} = \left\{ \sum_{\varphi_V} f(\varphi_V) \exp(-\beta H(\varphi_V | \bar{\varphi}_{V^c})) \right\} \Big/ \Xi(V | \bar{\varphi}_{V^c}) \tag{4}$$

where $\beta > 0$ is the inverse temperature.

Denote by $\varphi^{(k)}$, $k \in \mathbb{Z}^+$, the constant configuration $\varphi_x^{(k)} \equiv k$. Then:

Theorem DM.[4] There exist a constant β_0 and a sequence of continuous functions

$$\infty \equiv h_0^*(\beta) > h_1^*(\beta) > \cdots > h_k^*(\beta) > \cdots$$

such that for any $\beta \geq \beta_0$ and $k \in \mathbb{Z}^+$:

(i) In the interval $h \in (h_k^*(\beta), h_{k-1}^*(\beta))$ the model has a unique $\mathbb{Z}^d$-periodic Gibbs state generated by the boundary condition $\varphi^{(k)}$ and $\log \Xi(V | \varphi_{V^c}^{(k)})$ admits a convergent polymer expansion.

(ii) For $h = h_k^*(\beta)$ the set of $\mathbb{Z}^d$-periodic extreme limit Gibbs states consists precisely of two elements, generated by the boundary conditions $\varphi^{(k)}$ and $\varphi^{(k+1)}$; both $\log \Xi(V | \varphi_{V^c}^{(k)})$ and $\log \Xi(V | \varphi_{V^c}^{(k+1)})$ admit a convergent polymer expansion.

An improvement of this result was given in ref. 3.

Theorem CM.[3] For $d = 2$ there exists β_0 such that for all $\beta \geq \beta_0$ there are positive numbers $\{h_k^*(\beta)\}_{k=1}^{k_{\max}}$, with $k_{\max} = \lfloor \exp(\beta/20{,}000) \rfloor$, such that the following hold for $k = 1, \ldots, k_{\max}$:

(i) $\frac{1}{4}e^{-4\beta k} \leqslant \beta h_k^*(\beta) \leqslant 4e^{-4\beta k}$.

(ii) If $h_k^*(\beta) < h < h_{k-1}^*(\beta)$ [define $h_0^*(\beta) = +\infty$], then (a) there exists a unique Gibbs measure for the interaction (1), and (b) there exists $C_0(\beta, h) > 0$ such that for any $N \geqslant \lfloor 8/h + 1 \rfloor$ and cubic domain V of linear size N

$$\sup_{\bar{\varphi}_{V^c}, \bar{\varphi}'_{V^c}} |\langle \varphi_0 \rangle_{V, \bar{\varphi}_{V^c}} - \langle \varphi_0 \rangle_{V, \bar{\varphi}'_{V^c}}| \leqslant e^{-C_0(\beta, h) N}$$

(iii) If $h = h_k^*(\beta)$, then both $\log \Xi(V | \varphi_{V^c}^{(k)})$ and $\log \Xi(V | \varphi_{V^c}^{(k+1)})$ admit a convergent cluster expansion. Hence there are at least two distinct extreme Gibbs measures.

Here and below, $\lfloor \cdot \rfloor$ denotes the integer part. The improvement of ref. 3 over ref. 4 is the global uniqueness statement (ii.a) and the estimate of the dependence on the boundary condition (ii.b). Here we prove the following theorem.

Theorem. There exist a constant β_0 and a sequence of continuous functions

$$\infty \equiv h_0^*(\beta) > h_1^*(\beta) > \cdots > h_k^*(\beta) > \cdots$$

such that for any $\beta \geqslant \beta_0$ and $k \in \mathbb{Z}^+$:

(i) In the interval $h \in (h_k^*(\beta), h_{k-1}^*(\beta))$ the model has a unique limit Gibbs state generated by the boundary condition $\varphi^{(k)}$ and $\log \Xi(V | \varphi_{V^c}^{(k)})$ admits a convergent polymer expansion; moreover, there exists $C_1(\beta, h) > 0$ such that for any cube V of the linear size $N \geqslant \lfloor 4d/h + 1 \rfloor$

$$\sup_{\bar{\varphi}_{V}, \bar{\varphi}'_{V}} |\langle \varphi_0 \rangle_{V, \bar{\varphi}_{V^c}} - \langle \varphi_0 \rangle_{V, \bar{\varphi}'_{V^c}}| \leqslant e^{-C_1(\beta, h) N}$$

(ii) For $h = h_k^*(\beta)$ the set of $\mathbb{Z}^d$-periodic extreme limit Gibbs states consists precisely of two elements, generated by the boundary conditions $\varphi^{(k)}$ and $\varphi^{(k+1)}$; both $\log \Xi(V | \varphi_{V^c}^{(k)})$ and $\log \Xi(V | \varphi_{V^c}^{(k+1)})$ admit a convergent polymer expansion.

The difference between our theorem and one of ref. 3 is that we remove the restrictions $k \leqslant k_{\max}$ and $d = 2$. Comparing the methods of refs. 4 and 3 and the present paper one may say the following. In ref. 4 the authors constructed a polymer expansion for $\log \Xi(V | \varphi_{V^c}^{(k)})$ providing a complete proof for the case

$$e^{-4\beta k + \beta/100} \leqslant \beta h \leqslant e^{-4\beta(k-1) - \beta/100}$$

The rest of the proof in ref. 4 is rather sketchy and, as was pointed out in ref. 3, contains a mistake. This mistake is corrected here in the Appendix. Many details skipped in ref. 4 can be found in ref. 3.

The main achievement of ref. 3 is the control of a partition function with an *arbitrary* boundary condition. An arbitrary boundary condition produces a complicated boundary layer called a *boundary contour* which penetrates into the interior of V (see Section 8 for a precise definition). It was proven in ref. 3 that this boundary contour does not penetrate too deeply inside the domain. The result required a very detailed investigation of the geometry of this contour.

In the present paper we combine some results and ideas in refs. 4 and 3 with FKG inequalities, which permits us to avoid the study of this complicated boundary contour for an arbitrary boundary condition. Instead we show that global uniqueness follows from uniqueness in the class of translation-invariant limit Gibbs states, which is much easier to establish. In addition we develop a new method for analyzing the boundary contour which requires only a minimal knowledge of its geometry. What one really needs is "good control" over the Gibbs state with *stable* boundary conditions $\varphi^{(k)}_{V^c}$ [for $h \in (h^*_k(\beta), h^*_{k-1}(\beta))$] and rough *a priori* estimates on the influence of *unstable* sites, $\varphi_{x'} \neq k$, in the boundary condition, on the probability of the boundary contour. This good control is a simple consequence of the cluster expansion constructed in ref. 4 and *a priori* bounds are deduced from the Jensen and FKG inequalities. Remaining statements of our Theorem are simply borrowed from Theorem DM.

If instead of h one considers $\tilde{h} = \beta h$ as a parameter independent of β, then it is not hard to check that the analog of the above Theorem describing the low-temperature phase diagram of the model is true. In the parameter space $(\beta, \tilde{h})$ the high-temperature behavior of the model can also be understood: given $\tilde{h}$, there exists $\beta_1 = \beta_1(\tilde{h})$ such that for $\beta < \beta_1$ the limit Gibbs state is unique and admits a high-temperature expansion. This high-temperature limit Gibbs measure can be easily constructed as a small perturbation of the noninteracting system. Consequently each of the curves of the first-order phase transition constructed in the Theorem terminates at some point $\beta = \beta^{(k)}$ as β goes from ∞ to 0.

The situation in the parameter plane (β, h) is more complicated. We believe that for β small enough one still has a unique limit Gibbs state, but this measure cannot be considered as a small perturbation of an independent field. There may also be a difference in the high-temperature behavior of the model for $d = 2$ and $d \geqslant 3$, as is the case for the unconstrained SOS model without a field.[2,7]

There exist many other models of the SOS type (see, e.g., refs. 1 and 5). We believe that results similar to our Theorem are true for the class of models defined by the general formal Hamiltonian

$$H(\varphi)=\sum_{(x,x')} U(|\varphi_x-\varphi_{x'}|)+\sum_x G(\varphi_x)$$

where $U(\cdot)$ and $G(\cdot)$ are convex increasing functions from $\mathbb{R}^+$ to $\mathbb{R}^+$.

2. AN EQUIVALENT MODEL

It is clear that the state $\langle\cdot\rangle_{V,\bar{\varphi}_{V^c}}$ does not change if we replace the RHS of (2) by

$$\sum_{(x,x'):\, x,x'\in V} |\varphi_x-\varphi_{x'}|+\sum_{(x,x'):\, x\in V, x'\in V^c} (|\varphi_x-\bar{\varphi}_{x'}|-\bar{\varphi}_{x'})+h\sum_{x\in V}\varphi_x \quad (5)$$

Since

$$|\varphi_x-\bar{\varphi}_{x'}|-\bar{\varphi}_{x'}=\begin{cases}-\varphi_x & \text{for}\quad \varphi_x<\bar{\varphi}_{x'}\\ \varphi_x-2\bar{\varphi}_{x'} & \text{for}\quad \varphi_x\geqslant\bar{\varphi}_{x'}\end{cases} \quad (6)$$

it is natural, following ref. 3, to introduce the boundary condition $\bar{\varphi}_{V^c}\equiv+\infty$ setting

$$H(\varphi_V|\infty)=\sum_{(x,x'):\, x,x'\in V}|\varphi_x-\varphi_{x'}|-\sum_{(x,x'):\, x\in V,x'\in V^c}\varphi_x+h\sum_{x\in V}\varphi_x \quad (7)$$

and the boundary condition $\bar{\varphi}_{V^c}\equiv 0$ setting

$$H(\varphi_V|0)=\sum_{(x,x'):\, x,x'\in V}|\varphi_x-\varphi_{x'}|+\sum_{(x,x'):\, x\in V,x'\in V^c}\varphi_x+h\sum_{x\in V}\varphi_x \quad (8)$$

One can also consider a generalized boundary condition $\bar{\varphi}_{V^c}$ with $\bar{\varphi}_{x'}\in\{0\}\cup\mathbb{Z}^+\cup\{\infty\}$, $x'\in V^c$. In this case

$$\begin{aligned}H(\varphi_V|\bar{\varphi}_{V^c})=&\sum_{(x,x'):\, x,x'\in V}|\varphi_x-\varphi_{x'}|+h\sum_{x\in V}\varphi_x\\&+\sum_{\substack{(x,x'):\, x\in V,x'\in V^c\\ \bar{\varphi}_{x'}\neq 0,\infty}}(|\varphi_x-\bar{\varphi}_{x'}|-\bar{\varphi}_{x'})\\&+\sum_{\substack{(x,x'):\, x\in V,x'\in V^c\\ \bar{\varphi}_{x'}=0}}\varphi_x-\sum_{\substack{(x,x'):\, x\in V,x'\in V^c\\ \bar{\varphi}_{x'}=\infty}}\varphi_x\end{aligned} \quad (9)$$

and from now on we consider (9) as the definition of $H(\varphi_V|\bar{\varphi}_{V^c})$.

3. THE FINITENESS OF THE PARTITION FUNCTION

For $\bar{\varphi}_{V^c}$ with $\bar{\varphi}_{x'} \in \{0\} \cup \mathbb{Z}^+$, $x' \in V^c$, the finiteness of $\Xi(V \mid \bar{\varphi}_{V^c})$ is evident. The proof for the general case is given in Proposition 3.1 of ref. 3 and for the convenience of the reader we reproduce it below.

For $x \in \mathbb{Z}^d$ denote by $Q(x)$ some cube of the linear size $\lfloor (4d/h) + 1 \rfloor$ containing x. Consider a domain V such that for any $x \in V$ there exists $Q(x) \subseteq V$. Decompose $H(\varphi_V \mid \infty)$ into the sum $\sum_{i=1}^{d} H^{(i)}(\varphi_V \mid \infty)$ with

$$H^{(i)}(\varphi_V \mid \infty) = \sum_{(x,x'):\, x \in V,\, x' \in V}^{(i)} |\varphi_x - \varphi_{x'}| - \sum_{(x,x'):\, x \in V,\, x' \in V^c}^{(i)} \varphi_x + \frac{h}{d} \sum_{x \in V} \varphi_x \tag{10}$$

where both $\sum^{(i)}$ are taken only over (x, x') parallel to the ith coordinate axis. By the Schwarz inequality

$$\sum_{\varphi_V} \exp\left[-\beta \sum_{i=1}^{d} H^{(i)}(\varphi_V \mid \infty) \right] \leqslant \prod_{i=1}^{d} \left\{ \sum_{\varphi_V} \exp[-2\beta H^{(i)}(\varphi_V \mid \infty)] \right\}^{1/2} \tag{11}$$

Every factor $\sum_{\varphi_V} \exp[-2\beta H^{(i)}(\varphi_V \mid \infty)]$ is a product of one-dimensional partition functions of the form

$$\sum_{\varphi_L} \exp[-2\beta H^{(i)}(\varphi_L \mid \infty)] \tag{12}$$

where $L \subset \mathbb{Z}$ is a segment of the length greater than $\lfloor (4d/h) + 1 \rfloor$. For $L = [x', x'']$ and $\tilde{x}$ such that $\varphi_{\tilde{x}} = \min_{x \in L} \varphi_x$ we have

$$\begin{aligned} H^{(i)}(\varphi_L \mid \infty) &= -\varphi_{x'} + \sum_{x=x'}^{\tilde{x}-1} |\varphi_x - \varphi_{x+1}| \\ &\quad + \sum_{x=\tilde{x}}^{x''-1} |\varphi_x - \varphi_{x+1}| - \varphi_{x''} + \frac{h}{d} \sum_{x=x'}^{x''} \varphi_x \\ &\geqslant -\varphi_{x'} + |\varphi_{x'} - \varphi_{\tilde{x}}| + |\varphi_{\tilde{x}} - \varphi_{x''}| - \varphi_{x''} + \frac{h}{d} \sum_{x=x'}^{x''} \varphi_x \\ &= -2\varphi_{\tilde{x}} + \frac{h}{d} \sum_{x=x'}^{x''} \varphi_x \end{aligned}$$

$$\geqslant\left(\frac{h}{2d}|L|-2\right)\varphi_{\bar{x}}+\frac{h}{2d}\sum_{x=x'}^{x''}\varphi_x$$

$$\geqslant\frac{h}{2d}\sum_{x=x'}^{x''}\varphi_x \tag{13}$$

where the last inequality holds for $(h/2d)\,|L|\geqslant 2$, e.g., $|L|\geqslant\lfloor(4d/h)+1\rfloor$.

The finiteness of $\Xi(V|\infty)$ implies the finiteness of $\Xi(V|\bar{\varphi}_{V^c})$ for an arbitrary generalized $\bar{\varphi}_{V^c}$ because of the estimate

$$H(\varphi_V|\infty)\leqslant H(\varphi_V|\bar{\varphi}_{V^c})\leqslant H(\varphi_V|0) \tag{14}$$

which is a consequence of

$$-\varphi_x\leqslant|\varphi_x-\bar{\varphi}_{x'}|-\bar{\varphi}_{x'}\leqslant\varphi_x, \qquad \varphi_x,\varphi_{x'}\in\mathbb{Z}^+ \tag{15}$$

4. THE FKG INEQUALITIES

Introduce a partial order, $\leqslant$, on the configuration space by setting $\varphi_V\leqslant\varphi'_V$ iff $\varphi_x\leqslant\varphi'_x$ for all $x\in V$. A function $f(\varphi_V)$ is called *increasing* if $\varphi_V\leqslant\varphi'_V$ implies $f(\varphi_V)\leqslant f(\varphi'_V)$. The FKG inequalities[6] say that

$$\langle g(\varphi_V)\,f(\varphi_V)\rangle_{V,\bar{\varphi}_{V^c}}\geqslant\langle g(\varphi_V)\rangle_{V,\bar{\varphi}_{V^c}}\langle f(\varphi_V)\rangle_{V,\bar{\varphi}_{V^c}} \tag{16}$$

for any increasing $f(\varphi_V)$ and $g(\varphi_V)$ and for any boundary condition $\bar{\varphi}_{V^c}$. For our model these inequalities are true because (9) satisfies the Holley condition[8]

$$H(\varphi_V|\bar{\varphi}_{V^c})+H(\varphi'_V|\bar{\varphi}_{V^c})\geqslant H(\varphi_V\vee\varphi'_V|\bar{\varphi}_{V^c})+H(\varphi_V\wedge\varphi'_V|\bar{\varphi}_{V^c}) \tag{17}$$

where $\varphi_V\vee\varphi'_V=\max(\varphi_x,\varphi'_x)$ and $\varphi_V\wedge\varphi'_V=\min(\varphi_x,\varphi'_x)$.

5. MONOTONICITY IN THE BOUNDARY CONDITION AND THE VOLUME

A standard consequence of the FKG inequalities is a monotonicity in the boundary condition[9]:

$$\langle f(\varphi_V)\rangle_{V,\bar{\varphi}_{V^c}}\leqslant\langle f(\varphi_V)\rangle_{V,\bar{\varphi}'_{V^c}} \tag{18}$$

for $\bar{\varphi}_{V^c}\leqslant\bar{\varphi}'_{V^c}$ and increasing $f(\varphi_V)$. This implies that the states $\langle\cdot\rangle_{V,0}$ and $\langle\cdot\rangle_{V,\infty}$ are extreme in the sense

$$\langle f(\varphi_V)\rangle_{V,0}\leqslant\langle f(\varphi_V)\rangle_{V,\bar{\varphi}_{V^c}}\leqslant\langle f(\varphi_V)\rangle_{V,\infty} \tag{19}$$

For these extremal states another standard consequence of the FKG inequality is a monotonicity in the volume[9]:

$$\langle f(\varphi_V)\rangle_{V',0} \leqslant \langle f(\varphi_V)\rangle_{V'',0} \tag{20}$$

and

$$\langle f(\varphi_V)\rangle_{V',\infty} \geqslant \langle f(\varphi_V)\rangle_{V'',\infty} \tag{21}$$

for any increasing $f(\varphi_V)$ and $V \subseteq V' \subset V''$.

Consider now the bounded increasing functions

$$\sigma_x^m = \begin{cases} \varphi_x/m & \text{for} \quad \varphi_x < m \\ 1 & \text{for} \quad \varphi_x \geqslant m \end{cases} \tag{22}$$

It is not hard to see that the characteristic function $I_{\{\varphi_x = m\}}$ is given by

$$I_{\{\varphi_x=1\}} = 2\sigma_x^1 - 2\sigma_x^2, \quad I_{\{\varphi_x=m\}} = 2m\sigma_x^m - (m-1)\,\sigma_x^{m-1} - (m+1)\,\sigma_x^{m+1} \tag{23}$$

Hence the correlation functions

$$\left\langle \prod_{x\in A} \sigma_x^{m_x} \right\rangle_{V,\bar\varphi_{V^c}}, \qquad A \subseteq V \tag{24}$$

completely define the state $\langle\cdot\rangle_{V,\bar\varphi_{V^c}}$ and monotonicity in volume implies in a standard way the existence and translation invariance of the states $\langle\cdot\rangle_\infty$ and $\langle\cdot\rangle_0$.

6. ESTIMATES OF PROBABILITIES

This section contains a key estimate which allows us to handle the unboundedness of the spin variable. For models with bounded spin variables the difference in the energy of a configuration with one boundary condition and the energy of the same configuration with another boundary condition different from the first one at n sites is of order const $\cdot\, n$. Thus for any event its probability with the second boundary condition does not exceed that with the first boundary condition multiplied by the factor $\exp(2\beta \cdot \text{const} \cdot n)$. This is a rough but useful estimate which is clearly not true in our model, for an arbitrary event, because of the unboundedness of the spins. Nevertheless the estimate is still true for some events and these events appear to be sufficient for our purposes. To describe them denote by ψ_V a configuration from the set $\{-1, 0, 1\}^V$. Fix $k \in \mathbb{Z}^+$. Given ψ_V, we say that φ_V belongs to the class ψ_V if $\text{sign}(\varphi_x - k) = \psi_x$ for all $x \in V$. Clearly the class $\psi_V(\varphi_V)$ of the configuration φ_V is uniquely defined. Now we introduce the partition function

$$\Xi(V\,|\,\bar{\varphi}_{V^c}, \psi_V) = \sum_{\varphi_V \in \psi_V} \exp[-\beta H(\varphi_V\,|\,\bar{\varphi}_{V^c})] \tag{25}$$

and the corresponding state

$$\langle f(\varphi_V)\rangle_{V,\bar{\varphi}_{V^c},\psi_V} = \left\{ \sum_{\varphi_V \in \psi_V} f(\varphi_V) \exp[-\beta H(\varphi_V\,|\,\bar{\varphi}_{V^c})] \right\} \Big/ \Xi(V\,|\,\bar{\varphi}_{V^c}, \psi_V) \tag{26}$$

The difference between (25)–(26) and (3)–(4) is only in the free measure. Hence the measure $\langle\cdot\rangle_{V,\bar{\varphi}_{V^c},\psi_V}$ still satisfies the FKG inequalities and the results of Section 5 are still valid for this measure.

It is not hard to check that for large enough domains V, of the type considered in Section 3,

$$\langle \varphi_x\rangle_{V,\bar{\varphi}_{V^c},\psi_V} \leqslant C_2(k, \beta, h) \tag{27}$$

and

$$\langle \varphi_x\rangle_{V,\bar{\varphi}_{V^c}} \leqslant C_3(\beta, h) \tag{28}$$

(cf. Proposition 3.2 of ref. 3). Indeed, given $x \in V$, take a cube $Q(x) \subseteq V$. By the monotonicity in the boundary condition and the volume, we have

$$\begin{aligned}
&\langle \varphi_x\rangle_{V,\bar{\varphi}_{V^c},\psi_V}\\
&\quad\leqslant \langle \varphi_x\rangle_{V,\infty,\psi_V}\\
&\quad\leqslant \langle \varphi_x\rangle_{Q(x),\infty,\psi_V}\\
&\quad= \left\{ \sum_{\varphi_{Q(x)} \in \psi_{Q(x)}} \varphi_x \exp\left[-\beta \sum_{i=1}^{d} H^{(i)}(\varphi_{Q(x)}\,|\,\infty)\right]\right\} \Big/ \Xi(Q(x)\,|\,\infty, \psi_{Q(x)})
\end{aligned} \tag{29}$$

An argument similar to one used in Section 2 shows that the denominator of (29) is finite and greater than 1, while the numerator does not exceed

$$\begin{aligned}
&\left[\sum_{\varphi_x \in L(\psi_x)} \varphi_x \exp\left(-\frac{\beta h}{2d} \varphi_x \right)\right]\\
&\qquad \times \prod_{x' \in Q(x)\backslash x} \sum_{\varphi_{x'} \in L(\psi_{x'})} \exp\left(-\frac{\beta h}{2d} \varphi_{x'} \right) \leqslant C_2(k, \beta, h)
\end{aligned} \tag{30}$$

where

$$L(\psi_x)=\begin{cases}[1,k-1] & \text{for} \quad \psi_x=-1\\ k & \text{for} \quad \psi_x=0\\ [k-1,\infty) & \text{for} \quad \psi_x=1\end{cases} \tag{31}$$

Similarly one establishes the estimate (28).

Given any generalized boundary condition $\bar{\varphi}_{V^c}$, define other boundary conditions $\bar{\varphi}^0_{V^c}$ and $\bar{\varphi}^\infty_{V^c}$ as

$$\bar{\varphi}^0_{V^c}=\begin{cases}0 & \text{if} \quad \bar{\varphi}_x\neq k\\ k & \text{if} \quad \bar{\varphi}_x=k\end{cases} \tag{32}$$

and

$$\bar{\varphi}^\infty_{V^c}=\begin{cases}\infty & \text{if} \quad \bar{\varphi}_x\neq k\\ k & \text{if} \quad \bar{\varphi}_x=k\end{cases} \tag{33}$$

Denote by $I_{\psi_V}(\varphi_V)$ the indicator function of the event $\varphi_V\in\psi_V$ and introduce a shorthand notation

$$\langle\cdot\rangle_{V,k}=\langle\cdot\rangle_{V,\varphi^{(k)}_{V^c}}$$

Then by the monotonicity in the boundary condition

$$\begin{aligned}\frac{\langle I_{\psi_V}(\varphi_V)\rangle_{V,k}}{\langle I_{\psi_V}(\varphi_V)\rangle_{V,\bar{\varphi}_{V^c}}}&=\frac{\Xi(V\mid\varphi^{(k)}_{V^c},\psi_V)}{\Xi(V\mid\bar{\varphi}_{V^c},\psi_V)}\cdot\frac{\Xi(V\mid\bar{\varphi}_{V^c})}{\Xi(V\mid\varphi^{(k)}_{V^c})}\\ &\geqslant\frac{\Xi(V\mid\varphi^{(k)}_{V^c},\psi_V)}{\Xi(V\mid\bar{\varphi}^\infty_{V^c},\psi_V)}\cdot\frac{\Xi(V\mid\bar{\varphi}^0_{V^c})}{\Xi(V\mid\varphi^{(k)}_{V^c})}\end{aligned} \tag{34}$$

Obviously the second ratio is not less than

$$\exp\left(-2\beta\sum_{(x,x'):\,x\in V,x'\in V^c,\,\bar{\varphi}_{x'}\neq k}k\right) \tag{35}$$

The first ratio can be estimated as

$$\begin{aligned}&\frac{\Xi(V\mid\varphi^{(k)}_{V^c},\psi_V)}{\Xi(V\mid\bar{\varphi}^\infty_{V^c},\psi_V)}\\ &\quad=\left\langle\exp\left(-\beta\sum_{(x,x'):\,x\in V,x'\in V^c,\bar{\varphi}_{x'}\neq k}(|\varphi_x-k|-k+\varphi_x)\right)\right\rangle_{V,\bar{\varphi}^\infty_{V^c},\psi_V}\\ &\quad\geqslant\left\langle\exp\left(-2\beta\sum_{(x,x'):\,x\in V,x'\in V^c,\bar{\varphi}_{x'}\neq k}\varphi_x\right)\right\rangle_{V,\bar{\varphi}^\infty_{V^c},\psi_V}\end{aligned} \tag{36}$$

By the Jensen inequality the last expectation is greater than

$$\exp\left(-2\beta \sum_{\substack{(x,x'):\, x\in V, x'\in V^c,\\ \bar{\varphi}_{x'}\neq k}} \langle \varphi_x \rangle_{V,\bar{\varphi}^{\infty}_{V^c},\psi_V}\right)$$
$$\geqslant \exp\left(-2\beta \sum_{\substack{(x,x'):\, x\in V, x'\in V^c,\\ \bar{\varphi}_{x'}\neq k}} C_2(k,\beta,h)\right) \tag{37}$$

From (34)–(37) we finally obtain

$$\langle I_{\psi_V}\rangle_{V,\bar{\varphi}_{V^c}} \leqslant \langle I_{\psi_V}\rangle_{V,k} \exp\left(2\beta \sum_{(x,x'):\, x\in V, x'\in V^c, \bar{\varphi}_{x'}\neq k} C_4(k,\beta,h)\right) \tag{38}$$

where $C_4(k,\beta,h) = C_2(k,\beta,h) + k$.

7. UNIQUENESS

From now on we set, for a given h $[h \neq h_n^*(\beta),\ n\in\mathbb{Z}^+]$, $k = -\lfloor[\log(\beta h)]/4\beta\rfloor$ and accordingly change $C_2(k,\beta,h)$, $C_4(k,\beta,h)$ to $C_2(\beta,h)$, $C_4(\beta,h)$. For an arbitrary limit Gibbs state $\langle\cdot\rangle$ the monotonicity in the boundary condition (19) implies

$$\left\langle \prod_{x\in A} \sigma_x^{m_x} \right\rangle_0 \leqslant \left\langle \prod_{x\in A} \sigma_x^{m_x} \right\rangle \leqslant \left\langle \prod_{x\in A} \sigma_x^{m_x} \right\rangle_\infty \tag{39}$$

and the limit Gibbs state is unique whenever $\langle\cdot\rangle_\infty = \langle\cdot\rangle_0$. Take a cube V of linear size N and for any ψ_V, denote by $\Omega(\psi_V)$ the union of the connected components of the set $\{x\in V\colon \psi_x\neq 0\}$ which are adjacent to V^c. For any configuration φ_V the set $\Omega(\varphi_V) = \Omega(\psi_V(\varphi_V))$ is called the boundary contour. Introduce the event $\mathscr{E} = \{\varphi_V\colon |\Omega(\varphi_V)| > N^{d-0.5}\}$. The only result from ref. 4 which we need for the uniqueness is the estimate

$$\langle I_{\mathscr{E}}\rangle_{V,k} \leqslant \exp[-C_5(\beta,h)\, N^{d-0.5}], \qquad C_5(\beta,h) > 0 \tag{40}$$

which is valid because of the polymer expansion for $\langle\cdot\rangle_{V,k}$ constructed in ref. 4. Since the event $\mathscr{E}$ is defined in terms of ψ_V,

$$\langle I_{\mathscr{E}}\rangle_{V,\bar{\varphi}_{V^c}} \leqslant \langle I_{\mathscr{E}}\rangle_{V,k} \exp\left(2\beta \sum_{(x,x'):\, x\in V, x'\in V^c, \bar{\varphi}_{x'}\neq k} C_4(\beta,h)\right)$$
$$\leqslant \exp[-C_6(\beta,h)\, N^{d-0.5}] \tag{41}$$

where $\bar{\varphi}_{V^c}$ is an arbitrary generalized boundary condition, $C_6(\beta, h) = 0.5C_5(\beta, h) - 2\beta C_4(\beta, h)/\sqrt{N}$ and N is large enough. Bound (41) implies in the standard way[11] the uniqueness in the class of $\mathbb{Z}^d$-periodic limit Gibbs states. As $\langle\cdot\rangle_0$ and $\langle\cdot\rangle_\infty$ are translation invariant, one obtains $\langle\cdot\rangle_0 = \langle\cdot\rangle_k = \langle\cdot\rangle_\infty$ and hence global uniqueness.

8. A SKETCH OF AN ALTERNATE PROOF OF THE UNIQUENESS

In this section we apply the method going back to refs. 9 and 10. Consider the increasing functions

$$f(\varphi_V) = \sum_{x \in A} \varphi_x - \prod_{x \in A} \sigma_x^{m_x} \tag{42}$$

Then by (19)

$$\left\langle \sum_{x \in A} \varphi_x - \prod_{x \in A} \sigma_x^{m_x} \right\rangle_0 \leqslant \left\langle \sum_{x \in A} \varphi_x - \prod_{x \in A} \sigma_x^{m_x} \right\rangle_\infty \tag{43}$$

or equivalently

$$0 \leqslant \left\langle \prod_{x \in A} \sigma_x^{m_x} \right\rangle_\infty - \left\langle \prod_{x \in A} \sigma_x^{m_x} \right\rangle_0 \leqslant \sum_{x \in A} (\langle \varphi_x \rangle_\infty - \langle \varphi_x \rangle_0) \tag{44}$$

where the positivity follows from (39). In particular, $\langle \varphi_0 \rangle_0 = \langle \varphi_0 \rangle_\infty$ implies $\langle\cdot\rangle_0 = \langle\cdot\rangle_\infty$ and the uniqueness of the limit Gibbs state.

Now we will show how the uniqueness can be deduced from (44) and the differentiability in h of the free energy of the system. Comparing the energies of the corresponding configurations, it is not hard to see that for any disjoint domains V' and V''

$$\Xi(V' \mid 0) \cdot \Xi(V'' \mid 0) \leqslant \Xi(V' \cup V'' \mid 0) \tag{45}$$

and

$$\Xi(V' \mid \infty) \cdot \Xi(V'' \mid \infty) \geqslant \Xi(V' \cup V'' \mid \infty) \tag{46}$$

Using standard subadditivity arguments, we find that (45) and (46) imply the existence of the free energies

$$\lim_{V \nearrow \mathbb{Z}^d} F(V, h \mid 0), \quad F(V, h \mid 0) = |V|^{-1} \log \Xi(V \mid 0) \tag{47}$$

and

$$\lim_{V \nearrow \mathbb{Z}^d} F(V, h \mid \infty), \quad F(V, h \mid \infty) = |V|^{-1} \log \Xi(V \mid \infty) \tag{48}$$

By Jensen's inequality and estimate (28)

$$\begin{aligned}
0 &\geqslant F(V, h \mid 0) - F(V, h \mid \infty) \\
&= \lim_{V \nearrow \mathbb{Z}^d} |V|^{-1} \log \left\langle \exp\left(-2\beta \sum_{(x,x'):\, x \in V, x' \in V^c} \varphi_x\right)\right\rangle_{V,\infty} \\
&\geqslant \lim_{V \nearrow \mathbb{Z}^d} |V|^{-1} \left\langle\left(-2\beta \sum_{(x,x'):\, x \in V, x' \in V^c} \varphi_x\right)\right\rangle_{V,\infty} \\
&\geqslant \lim_{V \nearrow \mathbb{Z}^d} |V|^{-1} \left(-2\beta \sum_{(x,x'):\, x \in V, x' \in V^c} C_3(\beta, h)\right) = 0
\end{aligned} \tag{49}$$

Hence the free energy

$$F(h) = \lim_{V \nearrow \mathbb{Z}^d} |V|^{-1} \log \Xi(V \mid \bar{\varphi}_{V^c}) \tag{50}$$

exists and does not depend on the boundary condition $\bar{\varphi}_{V^c}$.

The function $F(V, h \mid 0)$ is bounded, differentiable, and convex because

$$\frac{d}{dh} F(V, h \mid 0) = \beta\, |V|^{-1} \sum_{x \in V} \langle \varphi_x \rangle_{V,0} > 0 \tag{51}$$

and

$$\frac{d^2}{dh^2} F(V, h \mid 0) = \beta^2\, |V|^{-2} \sum_{x \in V} \sum_{x' \in V} (\langle \varphi_x \varphi_{x'} \rangle_{V,0} - \langle \varphi_x \rangle_{V,0} \langle \varphi_{x'} \rangle_{V,0}) \geqslant 0 \tag{52}$$

where the last estimate follows from the FKG. The same is true for $F(V, h \mid \infty)$ and by the monotonicity in the volume the RHS of (44) is less than

$$\frac{|A|}{|V|} \sum_{x \in V} (\langle \varphi_x \rangle_{V,\infty} - \langle \varphi_x \rangle_{V,0}) = \frac{|A|}{\beta} \frac{d}{dh} (F(V, h \mid \infty) - F(V, h \mid 0)) \tag{53}$$

Functions $F(V, h \mid 0)$ and $F(V, h \mid \infty)$ have a common limit $F(h)$ which is also convex. Hence as soon as $(d/dh)\, F(h)$ exists, the RHS of (53) tends to 0 as $V \nearrow \mathbb{Z}^d$, implying the global uniqueness.

The differentiability of $F(h)$ follows from ref. 4, where the convergent polymer series for $F(h)$, $h \in (h_k^*(\beta), h_{k-1}^*(\beta))$, was constructed. This series can be differentiated term by term, which gives the value of $(d/dh)\, F(h)$.

9. THE DEPENDENCE ON THE BOUNDARY CONDITION

For any domain V denote by δV its external boundary

$$\delta V = \{x' \in V^c \colon \exists (x, x'), x \in V\} \tag{54}$$

Clearly the state $\langle \cdot \rangle_{V, \bar{\varphi}_{V^c}}$ depends only on $\bar{\varphi}_{\delta V}$ and we freely use the notation $\langle \cdot \rangle_{V, \bar{\varphi}_{\delta V}}$.

Take a sufficiently large N, and let V_1, V_2 be cubes of linear size $2N$ and $4N$: all cubes considered in this section are assumed to be centered at the origin. Consider now a domain $V \supseteq V_2$ whose boundary δV may coincide in part with the boundary of a cube $V_2' \supseteq V_2$. We assume that the boundary condition $\bar{\varphi}_{V^c}$ is such that $\bar{\varphi}_{\delta V}$ equals $\varphi^{(k)}$ at all sites of δV which do not belong to $\delta V_2'$, where on the set $\delta V \cap \delta V_2'$, $\bar{\varphi}_x$ can differ from k on at most $\sqrt{N}$ lattice sites. Introduce the event $\mathscr{E}_0 = \{\varphi_V \colon \Omega(\varphi_V) \cap V_1 \neq \varnothing\}$. Then

$$\langle I_{\mathscr{E}_0}(\varphi_V) \rangle_{V, \bar{\varphi}_{\delta V}} \leqslant e^{-C_7(\beta, h) N} \tag{55}$$

To see this, denote, for $\varphi_V \in \mathscr{E}_0$, by $\Omega_i(\varphi_V)$ the connected components of $\Omega(\varphi_V)$. By the construction every $\Omega_i(\varphi_V)$ touches δV and there exists at least one component intersecting V_1. Without loss of generality we suppose that $\Omega_1(\varphi_V)$ is such a component. This leads to the estimate

$$\begin{aligned} \langle I_{\mathscr{E}_0}(\varphi_V) \rangle_{V, \bar{\varphi}_{\delta V}} &\leqslant e^{2\beta C_4(\beta, h) \sqrt{N}} \langle I_{\mathscr{E}_0}(\varphi_V) \rangle_{V, k} \\ &\leqslant e^{2\beta C_4(\beta, h) \sqrt{N}} \sqrt{N}\, e^{-C_5(\beta, h) N} [1 + C_8(\beta, h)]^{\sqrt{N}} \\ &\leqslant e^{-C_7(\beta, h) N} \end{aligned} \tag{56}$$

Here, in the first inequality, we used (38), reducing the problem to the calculation for the stable boundary condition $\varphi^{(k)}$. The second inequality follows in a standard way from the cluster expansion constructed for $\langle \cdot \rangle_{V,k}$ in ref. 4. Namely, there are at most $\sqrt{N}$ possibilities to choose the site of δV which is touched by Ω_1 and the sum of the *statistical weights*[4] of all possible Ω_1 touching this site is less than $\exp[-C_5(\beta, h)\, N]$. (The bound takes into account the fact that the diameter of Ω_1, and hence $|\Omega_1|$, is not less than N). The constant

$C_8(\beta, h)$ estimates the sum of the statistical weights of all possible Ω_i touching a given lattice site and $[1 + C_8(\beta, h)]^{\sqrt{N}}$ estimates the statistical weight of all possibilities to choose Ω_i, $i \neq 1$. The whole estimate makes use of the standard fact that the statistical weight of Ω is the product of the statistical weights of Ω_i and is an upper bound for the probability of Ω (for definitions and details see ref. 4). The third inequality is trivial for $C_7(\beta, h) = 0.5 C_5(\beta, h)$ and N large enough.

Denote by $\langle \varphi_0 | \mathscr{E}_0^c \rangle_{V, \bar{\varphi}_{V^c}}$ the expectation of φ_0 conditioned on the complement of $\mathscr{E}_0$. This means that there exists a cube $V_1' \supseteq V_1$ such that $\varphi_x = \varphi_x^{(k)}$ for all $x \in \delta V_1'$. It is then a standard consequence of the polymer expansion that for some constant $C_9(\beta, h)$

$$|\langle \varphi_0 | \mathscr{E}_0^c \rangle_{V, \bar{\varphi}_{V^c}} - \langle \varphi_0 \rangle_{V,k}| \leqslant e^{-C_9(\beta,h)N} \tag{57}$$

Together with (55) and (27), this gives us

$$\begin{aligned} |\langle \varphi_0 \rangle_{V, \bar{\varphi}_{V^c}} - \langle \varphi_0 \rangle_{V,k}| &= |\langle I_{\mathscr{E}_0} \rangle_{V, \bar{\varphi}_{V^c}} (\langle \varphi_0 | \mathscr{E}_0 \rangle_{V, \bar{\varphi}_{V^c}} - \langle \varphi_0 \rangle_{V,k}) \\ &\quad + \langle I_{\mathscr{E}_0^c} \rangle_{V, \bar{\varphi}_{V^c}} (\langle \varphi_0 | \mathscr{E}_0^c \rangle_{V, \bar{\varphi}_{V^c}} - \langle \varphi_0 \rangle_{V,k})| \\ &\leqslant e^{-C_7(\beta,h)N} C_2(\beta, h) + e^{-C_9(\beta,h)N} \\ &\leqslant e^{-C_{10}(\beta,h)N} \end{aligned} \tag{58}$$

where $C_{10}(\beta, h) = 0.5 \min(C_7(\beta, h), C_9(\beta, h))$ and N is large enough.

Now we extend (58) to a wider class of boundary conditions. Let V_3 be the cube of linear size $6N$. Consider a domain $V \supseteq V_3$ which is a scaledup version of V considered before and suppose that the boundary condition $\bar{\varphi}_{V^c}$ is such that $\bar{\varphi}_{\delta V}$ equals $\varphi^{(k)}$ except for at most $(\sqrt{N})^2$ lattice sites belonging to $\delta V \cap \delta V_3'$ (where, as before, $V_3' \supseteq V_3$ is a cube). Define $\delta_i = \Delta_i \backslash \Delta_{i+1}$, where Δ_i is a cube of the linear size $6N - 2i + 2$. Let $\Omega^{(i)}(\varphi_V)$ be a union of the connected components of the set $\{x \in V \backslash \Delta_{i+1} \colon \varphi_x \neq \varphi_x^{(k)}\}$ adjacent to V^c. Introduce disjoint events

$$\begin{aligned} \mathscr{E}_i = \{\varphi_V \colon &|\Omega^{(1)}(\varphi_V) \cap \delta_1| \geqslant \sqrt{N}, \ldots, \\ &|\Omega^{(i-1)}(\varphi_V) \cap \delta_{i-1}| \geqslant \sqrt{N}, |\Omega^{(i)}(\varphi_V) \cap \delta_i| < \sqrt{N}\} \end{aligned} \tag{59}$$

$$\mathscr{E}_c = \left(\bigcup_{i=1}^{N-1} \mathscr{E}_i \right)^c$$

If $\varphi_V \in \mathscr{E}_c$, then the boundary contour $\Omega(\varphi_V)$ contains at least $N\sqrt{N}$ sites. Hence one has, similar to (56) or (40), the following estimate for the probability of $\mathscr{E}_c$:

$$\begin{aligned}\langle I_{\mathscr{E}_c}(\varphi_V)\rangle_{V,\bar{\varphi}_{\delta V}} &\leqslant e^{2\beta C_4(\beta,h)(\sqrt{N})^2}\langle I_{\mathscr{E}_c}(\varphi_V)\rangle_{V,k}\\ &\leqslant e^{2\beta C_4(\beta,h)(\sqrt{N})^2} e^{-C_5(\beta,h)\sqrt{N}\,N}\\ &\leqslant e^{-C_7(\beta,h)\,N\sqrt{N}}\end{aligned} \tag{60}$$

For $\varphi_V \in \mathscr{E}_i$ one can apply (58) to the domain V^i such that $V^i \cup \delta V^i = (V \backslash \Omega_i(\varphi_V)) \cup \Delta_i$. This gives us the bound

$$|\langle \varphi_0 | \mathscr{E}_i\rangle_{V,\bar{\varphi}_{V^c}} - \langle \varphi_0\rangle_{V,k}| \leqslant e^{-C_{10}(\beta,h)N} \tag{61}$$

Combining (60), (61), and (27), we conclude

$$\begin{aligned}|\langle\varphi_0\rangle_{V,\bar{\varphi}_{V^c}} - \langle\varphi_0\rangle_{V,k}| &= \Bigg|\sum_{i=1}^{N-1} \langle I_{\mathscr{E}_i}\rangle_{V,\bar{\varphi}_{V^c}} (\langle\varphi_0|\mathscr{E}_i\rangle_{V,\bar{\varphi}_{V^c}} - \langle\varphi_0\rangle_{V,k})\\ &\quad + \langle I_{\mathscr{E}_c}\rangle_{V,\bar{\varphi}_{V^c}} (\langle\varphi_0|\mathscr{E}_c\rangle_{V,\bar{\varphi}_{V^c}} - \langle\varphi_0\rangle_{V,k})\Bigg|\\ &\leqslant e^{-C_{10}(\beta,h)N} + e^{-C_7(\beta,h)N\sqrt{N}} C_2(\beta,h)\\ &\leqslant 2e^{-C_{10}(\beta,h)N}\end{aligned} \tag{62}$$

Expression (62) is a version of (58) which is weaker by the factor 2 in the RHS, but is applicable to the wider class of boundary conditions containing up to $(\sqrt{N})^2$ unstable points instead of only $\sqrt{N}$ required for (58).

The argument leading from (58) to (62) can be iterated several times. For example, the next step is the following. We extend (62) to the wider class of boundary conditions. Let V_4 be the cube of the linear size $8N$. Consider a domain $V \supseteq V_4$ which is a scaledup version of V considered before and suppose that the boundary condition $\bar{\varphi}_{V^c}$ is such that $\bar{\varphi}_{\delta V}$ equals $\varphi^{(k)}$ except for at most $(\sqrt{N})^3$ lattice sites belonging to $\delta V \cap \delta V'_4$ (here $V'_4 \supseteq V_4$ is a cube). Define $\delta_i = \Delta_i \backslash \Delta_{i+1}$, where Δ_i is a cube of the linear size $8N - 2i + 2$. Let $\Omega^{(i)}(\varphi_V)$ be a union of the connected components of the set $\{x \in V \backslash \Delta_{i+1}\colon \varphi_x \neq \varphi_x^{(k)}\}$ adjacent to V^c. Introduce disjoint events

$$\begin{aligned}\mathscr{E}_i = \{\varphi_V\colon &|\Omega^{(1)}(\varphi_V) \cap \delta_1| \geqslant (\sqrt{N})^2, \dots, |\Omega^{(i-1)}(\varphi_V) \cap \delta_{i-1}| \geqslant (\sqrt{N})^2,\\ &|\Omega^{(i)}(\varphi_V) \cap \delta_i| < (\sqrt{N})^2\}\end{aligned} \tag{63}$$

$$\mathscr{E}_c = \left(\bigcup_{i=1}^{N-1} \mathscr{E}_i\right)^c$$

If $\varphi_V \in \mathscr{E}_c$, then the boundary contour $\Omega(\varphi_V)$ contains at least $N(\sqrt{N})^2$ sites. Hence one has the following estimate for the probability of $\mathscr{E}_c$:

$$\begin{aligned}\langle I_{\mathscr{E}_c}(\varphi_V)\rangle_{V,\bar{\varphi}_{\delta V}} &\leqslant e^{2\beta C_4(\beta,h)(\sqrt{N})^3}\langle I_{\mathscr{E}_c}(\varphi_V)\rangle_{V,k} \\ &\leqslant e^{2\beta C_4(\beta,h)(\sqrt{N})^3} e^{-C_5(\beta,h)(\sqrt{N})^2 N} \\ &\leqslant e^{-C_7(\beta,h)\,N(\sqrt{N})^2} \end{aligned} \tag{64}$$

which is obtained similarly to (56) or (40).

For $\varphi_V \in \mathscr{E}_i$ one can apply (62) to the domain V^i such that $V^i \cup \delta V^i = (V \backslash \Omega_i(\varphi_V)) \cup \Delta_i$. This gives us the bound

$$|\langle \varphi_0 | \mathscr{E}_i\rangle_{V,\bar{\varphi}_{V^c}} - \langle \varphi_0\rangle_{V,k}| \leqslant e^{-C_{10}(\beta,h)\,N} \tag{65}$$

Combining (64), (65), and (27), we conclude

$$\begin{aligned}|\langle \varphi_0\rangle_{V,\bar{\varphi}_{V^c}} - \langle \varphi_0\rangle_{V,k}| &= \Bigg| \sum_{i=1}^{N-1} \langle I_{\mathscr{E}_i}\rangle_{V,\bar{\varphi}_{V^c}} (\langle \varphi_0 | \mathscr{E}_i\rangle_{V,\bar{\varphi}_{V^c}} - \langle \varphi_0\rangle_{V,k}) \\ &\quad + \langle I_{\mathscr{E}_c}\rangle_{V,\bar{\varphi}_{V^c}} (\langle \varphi_0 | \mathscr{E}_c\rangle_{V,\bar{\varphi}_{V^c}} - \langle \varphi_0\rangle_{V,k})\Bigg| \\ &\leqslant 2e^{-C_{10}(\beta,h)\,N} + e^{-C_7(\beta,h)\,N(\sqrt{N})^2} C_2(\beta,h) \\ &\leqslant 3e^{-C_{10}(\beta,h)\,N}\end{aligned} \tag{66}$$

Expression (66) is a version of (58) which is weaker by the factor 3 in the RHS but is applicable to the wider class of boundary conditions containing $(\sqrt{N})^3$ unstable points instead of $\sqrt{N}$ for (58).

After $2d$ iterations one obtains that for a cube V of the linear size $4dN$ with boundary condition $\bar{\varphi}_{\delta V}$ containing not more than $(\sqrt{N})^{2d}$ unstable points, i.e., for an arbitrary $\bar{\varphi}_{V^c}$,

$$|\langle \varphi_0\rangle_{V,\bar{\varphi}_{V^c}} - \langle \varphi_0\rangle_{V,k}| \leqslant 2de^{-C_{10}(\beta,h)\,N} \tag{67}$$

which obviously implies

$$\sup_{\bar{\varphi}_{V^c},\bar{\varphi}'_{V^c}} |\langle \varphi_0\rangle_{V,\bar{\varphi}_{V^c}} - \langle \varphi_0\rangle_{V,\bar{\varphi}'_{V^c}}| \leqslant e^{-C_1(\beta,h)\,4dN} \tag{68}$$

for $C_1(\beta,h) = C_{10}(\beta,h)/8d$ and N large enough. This finishes the proof of the Theorem.

Note that an analog of (68) for any local function $f(\varphi_A)$ can be obtained either by a similar argument or directly from (68) and the finite-volume version of (44).

APPENDIX

To correct the mistake in ref. 4 one should replace the second paragraph on p. 560 of ref. 4 by the following.

First we construct $h_k^*(\beta)$. For this purpose we complete the set of elementary cylinders by all cylinders $\eta = (\tilde{\eta}, k+1, k)$ and $\eta = (\tilde{\eta}, k, k+1)$. Then for contours $\Gamma = \{\gamma^{\text{ext}}, \gamma_i, \gamma^{\text{int},j}\}$ we redefine the statistical weight of Γ as

$$\begin{aligned}\tilde{w}(\Gamma) = {} & Z^{-1}(\text{Supp}(\Gamma)^{(E(\Gamma),\dots)}) \\ & \times w(\gamma^{\text{ext}})\, Z(\text{Supp}_e(\Gamma)^{(I(\gamma^{\text{ext}}),\dots)}) \\ & \times \prod_i w(\gamma_i)\, Z(\text{Supp}_i(\Gamma)^{(I(\gamma_i),\dots)}) \\ & \times \prod_j w(\gamma^{\text{int},j}) \frac{\Xi(\bar{\gamma}^{\text{int},j(I(\gamma^{\text{int},j}),S(\gamma^{\text{int},j}))})}{\Xi(\bar{\gamma}^{\text{int},j(E(\Gamma),S(\gamma^{\text{int},j}))})}\end{aligned}$$

instead of (2.14). Obviously representation (2.33) remains valid for both $\Xi(V^{(k,\cdot)})$ and $\Xi(V^{(k+1,\cdot)})$, but we do not have good bounds for $\tilde{w}(\eta)$ and $\tilde{w}(\Gamma)$. Hence the cluster expansion for $\log \Xi(V^{(k,\cdot)})$ and $\log \Xi(V^{(k+1,\cdot)})$ cannot be written immediately. Instead of this we introduce the truncated statistical weights

$$\bar{w}(\eta) = \min(\tilde{w}(\eta), \exp\{-\tfrac{1}{3}\beta\, |\tilde{\eta}|\, L(\eta)\}) \tag{2.46a}$$

and

$$\bar{w}(\Gamma) = \min\left(\tilde{w}(\Gamma),\, w(\Gamma) \prod_j w(\gamma^{\text{int},j})^{-1} \exp\left\{-\tfrac{1}{3}\sum_j \beta\, |\tilde{\gamma}^{\text{int},j}|\, L(\gamma^{\text{int},j})\right\}\right) \tag{2.46b}$$

and the corresponding partition functions $\bar{\Xi}(V^{(k,\cdot)})$ and $\bar{\Xi}(V^{(k+1,\cdot)})$.

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